IV. Preliminary Investigation

In this section, certain results will be presented, which have been obtained in the preliminary investigation already undertaken. The proofs of the new theorems will be left for the Appendix.

The point of departure is a theorem due to Jackson [20] which has already been referred to. He considers a situation in which there are M departments, the m^{th} department having the following properties ($m = 1, 2, \ldots, M$):

1. N_m servers
2. Customers from outside the system arrive in a Poisson-type time series at mean rate λ_m (additional customers will arrive from other departments in the system).
3. Service is on a first come, first served basis, with an infinite storage available for overflow; the servicing time being exponentially distributed with mean $1/\mu_m$.
4. Once served, a customer goes immediately from department m to department k with probability θ_{km}; his total service is completed (and he then leaves the system) with probability $1 - \sum_k \theta_{km}$.

Property 4 is the basis on which Jackson calls this system a network of waiting lines. Defining Γ_m as the average arrival rate of customers at department m from all sources, inside and outside the system, Jackson states that

$$\Gamma_m = \lambda_m + \sum_k \theta_{mk} \cdot \Gamma_k \tag{1}$$

Now, defining n_m as the number of customers waiting and in service at department m, and defining the state of the system as the vector $(n_1, n_2, \ldots, n_M)$, he proves the following

THEOREM: Define $P_n^{(m)}$ ($m = 1, 2, \ldots, M$, $n = 0, 1, 2, \ldots$), the Pr [finding n customers in department m in the steady state], by the following equations (where the $P_0^{(m)}$ are determined by the conditions $\sum_n P_n^{(m)} = 1$):

$$P_n^{(m)} = \begin{cases} P_0^{(m)} \left(\Gamma_m / N_m \mu_m \right)^n N_m^{\ n} / n! & (n = 0, 1, \ldots, N_m) \\ P_0^{(m)} \left(\Gamma_m / N_m \mu_m \right)^n (N_m)^{N_m} / N_m! & n \geq N_m \end{cases} \tag{2}$$

A steady state distribution of the state of the above described system is given by the products

$$P(n_1, n_2, \ldots, n_M) = P_{n_1}^{(1)} P_{n_2}^{(2)} \ldots P_{n_M}^{(M)} \tag{3}$$

provided $\Gamma_m < \mu_m N_m$ for $m = 1, 2, \ldots, M$

This theorem says, in essence, that at least so far as steady states are concerned, the system with which we are concerned behaves as if its departments were independent elementary systems of the following type (which is the type considered by Erlang): Customers arrive in a Poisson type time series at mean rate λ. They are handled on a first come, first serve basis by a system of N identical servers, the servicing times being exponentially distributed with mean $1/\mu$. The steady state distribution of the number of people, n, waiting and in service has been obtained by Erlang, and is the identical form as in Jackson's theorem above, with $N_m = N$, $\Gamma_m = \lambda$, $\mu_m = \mu$, $P_n^{(m)} = P_n$, and with the condition $\lambda < \mu N$. That is, Jackson's problem reduces to that of Erlang's when $M = 1$. However, for $M = 1$, the network property of the system is destroyed. Jackson's result is very neat, and suggests the possibility of being able to handle large nets of the type of interest to this thesis.

Following, is a statement and discussion of some results obtained for systems similar to those considered by Erlang and Jackson; proofs for the theorems are given in the Appendix.

Consider a pair of nodes in a large communication net. When the first of these nodes transmits a message destined for the other, one can inquire as to what the rest of the net appears like, from the point of view of the transmitting node. In answer to this inquiry, it does not seem unreasonable to consider that the rest of the net offers, to the message, a number N, of "equivalent" alternate paths from the first node to the second; the equivalence being a very gross simplification of the actual situation, which, nevertheless, serves a useful purpose. Thus, the system under consideration reduces itself to that considered by Erlang. Now, for given conditions of average traffic flow and total transmitting capacity between the two nodes, the problem as to the optimum value of N presents itself (optimum here referring to that value of N which minimizes the total time spent in the transmitting node, i.e., time spent waiting for a free transmission channel plus time spent in transmitting the message). Thus, as shown in Figure 1, the system consists of N channels, each of capacity C/N bits per second, with Poisson arrivals of mean rate λ, and with the message lengths distributed exponentially with mean length $1/\mu$ bits.

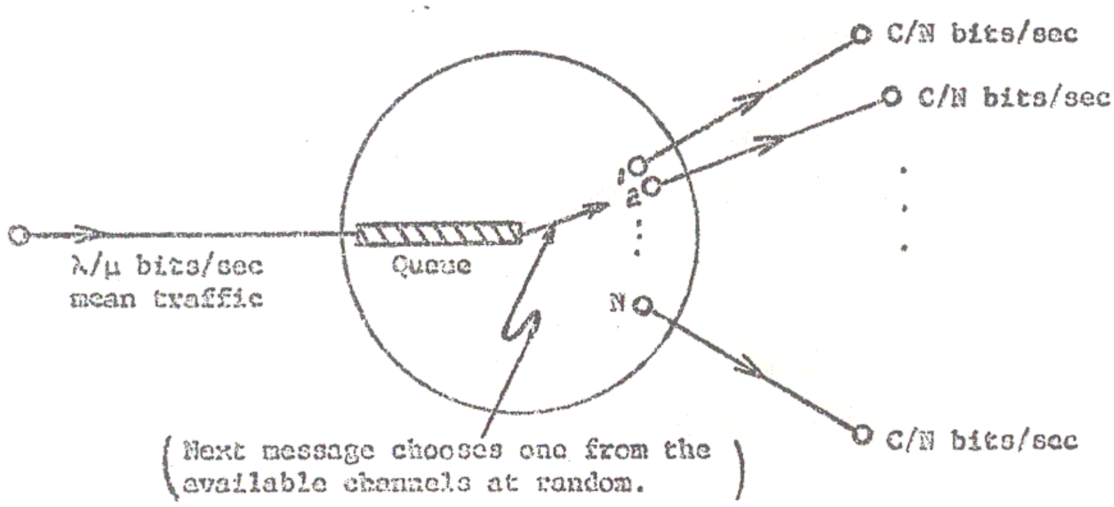

Figure 1: N-channel node considered in Theorem 1.

As is well known, the solution for P_n (defined as the probability of finding n messages in the system in the steady state) is, for $\lambda/\mu C < 1$,

$$P_n = \begin{cases} P_0 p^n N^n / n! & n \leqslant N \\ P_0 p^n N^N / N! & n \geqslant N \end{cases} \qquad (4)$$

where $p = \lambda/\mu C$ is defined as the utilization factor. Note that this is the same solution found by Erlang. From these steady state probabilities, we can easily find E(t), which is the expected value of the time spent in the system, as

$$E(t) = N/\mu C + P(\geqslant N) / \mu C(1-p) \qquad (5)$$

where

$$P(\geqslant N) = P_0 (Np)^N / (1-p)N!$$

and

$$P_0 = [\sum_{n=0}^{N-1} (Np)^n/n! + (Np)^N/(1-p)N!]^{-1}$$

We are now ready to state

THEOREM 1:

The value of N which minimizes E(t), for all $0 \leqslant p < 1$ is $N = 1$.

Let us look at the expression for E(t) a little closer. Note that the quantity $N/\mu C$ is merely the average time spent in transmitting the message over the channel, once a channel is available. Also, $P(\geqslant N)$ is the probability that a message is forced to enter the queue. Now, from the independence of the messages, one would expect E(t) to be

E(t) = average time spent in channel + average time spent in queue.

Equation (5) is of the form

E(t) = average time spent in channel + (probability of entering the queue)T

where $T = 1/(1-\rho)\mu C$.

The physical interpretation of the quantity T is that it is the average time spent in the queue, given that a message will join the queue. The interesting thing here is that the quantity T is independent of N.

Let us now recall one of the basic assumptions of Jackson's theorem, namely, that upon completing service in department m, a customer goes immediately to department k with probability π_{km}. If, now, we consider a communication network of nodes and links (channels), it is not at all obvious how we can route messages in the net so as to satisfy this assumption. That is, how can we design a communication network so that an arbitrary message entering node m will, with probability π_{km} be transmitted over that channel which links node m to node k. Clearly, one way to achieve this is to assign each message, as it enters node m, to the channel linking nodes m and k, with probability π_{km}. However, with such a scheme, there would occur situations in which there were messages in the node waiting on a queue at the same time that some of the channels leading out of the node were idle. It seems reasonable, in some cases at least, to prohibit such a condition. Therefore, restricting the existence of idle channels if there are any waiting messages, we arrive at the following

THEOREM 2:

Given a two channel service facility of total capacity C, Poisson arrivals with mean rate λ, message lengths distributed exponentially with mean length $1/\mu$, and the restriction that no channel be idle if a message is waiting in the queue, then, for an arbitrarily chosen number, $0 \leq \alpha_1 \leq 1$ it is <u>not</u> possible to find a queue discipline and an assignment of the two channel capacities (the sum being C) such that

Pr (entering message is transmitted on the first channel) $= \alpha_1$

for all $0 \leq p < 1$

where $p = \lambda/\mu C$

Thus, this theorem shows that one cannot, in general, make an arbitrary assignment of the probability of being transmitted over a particular channel which remains constant for all p. However, in the proof of this theorem, it is shown that it is possible to find a queue discipline and a channel capacity assignment such that the deviation of this probability α_1 is rather small over the entire range $0 \leq p < 1$.

It is also of interest to note that in the proof of Theorem 2, it is shown that the variation of α_1 is zero over $0 \leq p < 1$ for $\alpha_1 = 0, 1/2, 1$. In fact this leads to the following

<u>COROLLARY:</u> For the same conditions as Theorem 2, except allowing N channels, and for $\alpha_1 = \alpha_2 = \ldots = \alpha_N = 1/N$, then it is possible to find a queuing discipline and a channel capacity assignment such that

Pr (entering message is transmitted over the i^{th} channel) = $1/N$ for all $0 \leq p < 1$

In proving Theorems 4 and 5, as well as in some other investigations which have been started by the author, the solution to a set of non-linear equations was found to be necessary. As is sometimes possible with such equations, the proper transformation of variables permitted the reduction of these equations to a linear system. This transformation turned out to involve that fundamental quantity p, and thus led to

THEOREM 3:

Consider an N channel service facility of total capacity C, Poisson arrivals with mean rate λ, message lengths distributed exponentially with mean length $1/\mu$, and an arbitrary queue discipline. Define the utilization factor

$$p = \lambda/\mu C$$

Then

$$p = 1 - \sum_{n=0}^{\infty} (\overline{C_n}/C) P_n \tag{6}$$

where $\overline{C_n}$ = Expected value of the unused capacity given n lines in use

and P_n = Pr (finding n messages in the system in the steady state)

provided the system reaches a steady state.

Notice that, in Theorem 3, all information regarding the queue discipline is contained and summarized in the quantity $\overline{C_n}$. This theorem corresponds very nicely with one's intuition, as may be seen by rewriting it as

$$p = 1 - E \text{ (unused normalized capacity)}$$

where the normalization is with respect to the total capacity C. It is clear that this last equation may, in turn be written as

$$p = E \text{ (used normalized capacity)}$$

which says that

$$\lambda/\mu = E \text{ (used capacity)} \tag{7}$$

Now, since the average number of messages entering per second is λ and their average length is $1/\mu$ bits per message, the quantity λ/μ is clearly the average number of bits per second entering the facility. Recall that the condition for the existence of a steady state for this system is

$$\lambda/\mu C < 1$$

Thus, if we have a steady state solution, we are guaranteed that $\lambda/\mu < C$ (which says that the facility can handle the incoming traffic) and so the expected value of the capacity used by this input rate will merely be λ/μ; this is precisely what equation (7) states.

In even the simplest conceivable communications network, it seems reasonable to require that when a message reaches the node to which it is addressed, it should leave the system i.e., it is delivered. However, in the assumptions considered by Jackson, there is no final address associated with each "message" and so, the correspondence between the problem considered by Jackson, and that of interest to this thesis is not as close as one might hope.

Therefore, let us consider a communication network with N + 1 nodes, for which the entering messages have associated with them a final destination (address). Once a message reaches its address, it is dropped from the system immediately. Thus, we are altering the model considered by Jackson only slightly; and in order to keep the rest of the system similar to his, we will consider a completely connected net, with all $\pi_i = 1/N$ (i.e., upon entering a node, a message will be transmitted over a particular channel with probability 1/N, unless the node which it just entered is its final destination, in which case the message leaves the system with probability one). Note that the corollary to Theorem 2 allows us to define such π_i. For such a system, it turns out that Jackson's results still apply with some slight midifications, as stated in

<u>THEOREM 4</u>:

Consider the completely connected N + 1 node system described above. Let each transmission channel leaving node m have capacity C_m/N. Let the incoming messages entering node m from external sources be Poisson at rate λ_m and let the message lengths be exponentially distributed with mean length $1/\mu$. Further, let ε_{mj} be the Pr (message entering node m from its external source has, for a final address, node j). Also define $p_n^{(m)}$ as the probability of finding n messages in node m in the steady state.

Then

$$P_n^{(m)} = \begin{cases} P_0^{(m)} (\Gamma_m/\mu C_m)^n N^n/n! & (n = 0, 1, \ldots, N) \\ P_0^{(m)} (\Gamma_m/\mu C_m)^n N^N/N! & (n = N, N+1, \ldots) \end{cases} \quad (8)$$

where

$$\Gamma_m = \lambda_m + \sum_{i \neq m} \Gamma_i \ \alpha_{im}/N \quad (9)$$

and α_{im} = Pr (arbitrary message in node i does not have node m for a final address)

provided $\Gamma_m < \mu C_m$ for all $m = 1, 2, \ldots, M$.

This theorem is almost identical to Jackson's theorem, as one might expect. Notice that here, the appropriate definition for the utilization factor for node m is $\rho_m = \Gamma_m/\mu C_m$. The definition of Γ_m as given in Eqn. (9) can be shown to agree with the definition for the average arrival rate of messages at node m (analogous to Jackson's definition in Eqn. (1)). The evaluation of α_{im} involves solving a set of simultaneous equations, as does the evaluation of Γ_m. By way of illustration, the solution for Γ_1 and α_{12} in a three node net follows:

$$\Gamma_1 = \frac{2}{3} [2\lambda_1 + \lambda_2 t_{23} + \lambda_3 t_{32}]$$

$$\alpha_{12} = [2\lambda_1 t_{13} + \lambda_2 t_{23}]/(3/2) \Gamma_1$$

As already mentioned, P. Burke [17] has shown that in a waiting system with N servers, with Poisson arrivals (mean rate λ) and with exponential holding times (mean holding time for each server = 1/μ), the traffic departing from the system is Poisson with mean rate λ, providing the steady state prevails (i.e. provided $p = \lambda/\mu M$ is less than 1). In fact, it is on this basis that Jackson is able to say that his system consists of independent elementary systems; that is, Burke's theorem states that exponential waiting systems (or departments or nodes, as the problem may be defined) always transform Poisson input traffic into Poisson output traffic (with the same mean rates) and thus the departing traffic is not distinguishable from the input traffic. An identical situation exists for the system considered in Theorem 4, and is stated formally in

THEOREM 5:

For the system considered in Theorem 4, all traffic flowing within the network is Poisson in nature, and, in particular, the traffic transmitted from node m to any other node in the system in Poisson with mean rate Γ_m/N.

Many of the theorems presented here are fairly specialized to particular conditions on the network topology and on the routing discipline. It is anticipated that a number of them can be extended to less restrictive networks, and such an effort is now being undertaken by the author, since this investigation fits very well with the general aims of the thesis research.