APPENDIX

Proof of Theorems

Before we proceed with the proofs, let us derive a general result for a class of Birth-Death Processes [12].

Let[4]

$P_n(t)$ = Pr [finding n members in system at time t]

$b_n dt$ = Pr [birth of a new member during any interval of length dt | n members already in system]

$d_n dt$ = Pr [death of a member during any interval of length dt | n members in system]

then, clearly

$$P_0(t + dt) = P_1(t)\,(d_1 dt) + P_0(t)\,(1-b_0 dt)$$

$$P_n(t + dt) = P_{n+1}(t)\,(d_{n+1} dt) + P_{n-1}(t)\,(b_{n-1} dt) + P_n(t)\,(1-d_n dt - b_n dt) \qquad n \geq 1$$

From these eqns., we get

$$dP_0(t)/dt = d_1 P_1(t) - b_0 P_0(t) \qquad \text{(A1)}$$

$$dP_n(t)/dt = d_{n+1} P_{n+1}(t) + b_{n-1} P_{n-1}(t) - (d_n + b_n) P_n(t) \qquad n \geq 1 \qquad \text{(A2)}$$

Let us now assume the existence of a steady state distribution for $P_n(t)$, that is,

$$\lim_{t \to \infty} P_n(t) = P_n$$

Therefore $\lim_{t \to \infty} dP_n(t)/dt = 0$

and so, we get, for eqns. (A1) and (A2),

$$0 = d_1 P_1 - b_0 P_0$$

$$0 = d_{n+1} P_{n+1} + b_{n-1} P_{n-1} - (d_n + b_n) P_n \qquad n \geq 1$$

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[4] Note that b_n and d_n are assumed to be independent of time.

The solution to this set of difference equations is

$$P_n = \prod_{i=0}^{n-1} P_0(b_i/d_{i+1}) \qquad n \geq 1 \qquad \text{(A3)}$$

which may easily be checked.

<u>Theorem 1 - proof:</u>

Given $\qquad E(t) = N/\mu C + P(\geq N)/\mu C(1-p)$

substituting for $P(\geq N)$ and rearranging terms gives us

$$E(t) = (N/\mu C)\ [1 + \frac{1/N(1-p)}{S_N(1-p) + 1}\ ] \qquad \text{(A4)}$$

where $\quad S_N = \sum_{n=0}^{N-1} (Np)^{n-N} N!/n! \ > \ 0$

now $\quad S_N = \sum_{n=0}^{N-1} p^{n-N}\ [N/N][(N-1)/N]\ \ldots\ [(n+1)/N]$

therefore $\quad S_N \leq \sum_{n=0}^{N-1} p^{n-N} = (p^{-N}-1)/(1-p)$

giving $\quad 0 < S_N \leq \frac{p^{-N}-1}{1-p} \qquad \text{(A5)}$

Now, for $N = 1$, eqn. (A4) yields

$$E(t) = 1/\mu C(1-p) \qquad \text{for } N = 1$$

therefore, it is sufficient to show that

$$E(t) > 1/\mu C(1-p) \text{ for all } N > 1, \quad 0 \leq p < 1$$

using (A5) we get, for (A4)

$$E(t) \geq (N/\mu C)\ [1 + p^N/N(1-p)]$$

$$E(t) \geq [N(1-p) + p^N]/\mu C(1-p)$$

Letting $1-p = \alpha$ or $1-\alpha = p$, we see that

$$N(1-p) + p^N = N\alpha + (1-\alpha)^N \geq N\alpha + 1-N\alpha = 1$$

thus $\quad E(t) \geq 1/\mu C(1-p)$

for all N, and in particular, the only case for which the equality holds is $N = 1$. Note that the equality would also hold for $\alpha = 0$ but this implies that $p = 1$, which we do not permit. Thus

$$E(t) > 1/\mu C(1-p) \quad \text{for } N > 1, \quad 0 \leq p < 1$$

which completes the proof. §

Theorem 2 - proof:

The method of proof will be to show the impossibility of contradicting the theorem.

Suppose $p \rightarrow 0$. Then P_0 (the probability that in the steady state the system is empty) approaches 1. In such a case, an entering message (which will, with probability arbitrarily close to 1, find an empty system) must be assigned to channel 1 with probability π_1 (and to channel 2 with probability $\pi_2 = 1-\pi_1$) if one is to have any hope of contradicting the theorem.

Now suppose $p \rightarrow 1$; then P_0 and P_1 (the probability of one message in the system) both approach 0. Therefore, the channel capacity C_1 assigned to channel 1 (which implies $C-C_1 = C_2$ for channel 2) must be chosen so that

$$\alpha \equiv \Pr\,[\text{channel 1 empties before channel 2}\,|\,\text{both channels busy}] = \pi_1$$

That is, with probability arbitrarily close to 1, a message entering the node will be forced to join a queue, and so, when it reaches the head of the queue, it will find both channels busy. If this message is to be transmitted over channel 1 with probability π_1, it must be that the channel capacity assignments result in $\alpha = \pi_1$. Note that we have taken advantage of the fact that messages with exponentially distributed lengths exhibit no memory as regards their transmission time.

Now, $$\alpha = \int_{t=0}^{t=\infty} \Pr[\text{channel 1 empties in } (t,t+dt) \mid \text{both busy at time 0}] \cdot \Pr[\text{channel 2 is not yet empty by } t \mid \text{both busy at time 0}]$$

$$\alpha = \int_0^{\infty} \mu C_1 e^{-\mu C_1 t} e^{-\mu C_2 t} dt$$

$$\alpha = \mu C_1/(\mu C_1+\mu C_2) = C_1/C$$

but

$$\alpha = \varkappa_1$$

therefore

$$C_1 = \varkappa_1 C$$

and also

$$C_2 = \varkappa_2 C = (1 - \varkappa_1)C$$

These two limiting cases for $p\to 0$ and $p\to 1$ have constrained the construction of our system completely.

Now, let

$r_1 = \Pr$ [incoming message is transmitted on channel 1]

$P_n = \Pr$ [finding n messages in the system in the steady state]

Then clearly,

$$r_1 = \varkappa_1 P_0 + q_{21} P_1 + \sum_{n=2}^{\infty} \varkappa_1 P_n \tag{A6}$$

where

$q_{i1} = \Pr$ [channel i is busy | only one channel is busy]

that is

$r_1 = E$ [probability of an arbitrary message being transmitted over channel 1]

For q_{21}, we write:

$$q_{21}(t+dt) = [P_0(t)/P_1(t)](\lambda\varkappa_2 dt) + [P_2(t)/P_1(t)](\mu C_1 dt) + q_{21}(t)[1-\lambda dt-\mu C_2 dt]$$

Assuming a steady state distribution, we get,

$$0 = (P_0/P_1)\lambda\varkappa_2 + (P_2/P_1)\mu C_1 - (\lambda+\mu C_2)q_{21} \tag{A7}$$

Now, since this system satisfies the hypothesis of the Birth-Death Process considered earlier, we apply Eqn.(A3), with $d_1 = \mu\overline{C}$, $d_n = \mu C$ $(n \geq 2)$, $b_n = \lambda$, and obtain

$$P_n = \begin{cases} (C/\bar{C})\rho^n P_0 & n \geq 1 \\ P_0 & n = 0 \end{cases} \qquad \text{(A8)}$$

where $\rho = \lambda/\mu C$,

and $\bar{C} = E$ [capacity in use | one channel is busy]

$$= \mu C_1 q_{11} + \mu C_2 q_{21}$$

Also, recall that $C_1 = \varkappa_1 C$ and $C_2 = \varkappa_2 C = (1-\varkappa_1)C$.
Thus, Eqn.(A7) becomes

$$q_{21} = (\mu\bar{C}\varkappa_2 + \lambda\varkappa_1) \; / \; (\lambda + \mu C\varkappa_2) \qquad \text{(A9)}$$

similarly

$$q_{11} = (\mu\bar{C}\varkappa_1 + \lambda\varkappa_2) \; / \; (\lambda + \mu C\varkappa_1)$$

Now, forming the equation,

$$q_{11} + q_{21} = 1$$

we obtain, after some algebra,

$$\mu\bar{C} = \mu C(\mu C + 2\lambda) \; / \; (2\mu C + \lambda/\varkappa_1\varkappa_2)$$

$$= \mu C(1 + 2\rho) \; / \; (2 + \rho/\varkappa_1\varkappa_2)$$

We may now write Eqn.(A9) as

$$q_{21} = \frac{[\mu C(1+2\rho)/(2+\rho/\varkappa_1\varkappa_2)]\varkappa_2 + \lambda\varkappa_1}{\lambda + \mu C\varkappa_2}$$

Simplifying, we get

$$q_{21} = \varkappa_1(\varkappa_2+\rho) \; / \; (2\varkappa_1\varkappa_2+\rho) \qquad \text{(A10)}$$

Returning to Eqn.(A6), we see that the only way in which r_1 can equal $\varkappa_1$ is for $q_{21} = \varkappa_1$. Eqn. (A10) shows that this is <u>not</u> the case, which demonstrates that the theorem cannot be contradicted for an arbitrary $\varkappa_1$, proving the theorem. ▯

However, it can be seen from Eqn. (A10), that $q_{21} = x_1$ for $x_1 = 0,\ 1/2,\ 1$ only. Let us now form r_1 from Eqns. (A6) and (A8):

$$r_1 = P_0 [\, x_1 + (\lambda q_{21}/\mu\bar{C}) + (x_1 C/\bar{C}) \sum_{n=2}^{\infty} p^n]$$

where P_0 is found from Eqn. (A8) by requiring

$$\sum_{n=0}^{\infty} P_n = 1$$

After substituting and simplifying, we get

$$r_1 = x_1 \left[\frac{x_1 p^2 + (1-x_1^{\,2})p + x_1 x_2}{(1-2x_1 x_2)p^2 + 3x_1 x_2 p + x_1 x_2} \right]$$

Figure (A-1) shows a plot of r_1 as a function of p, with x_1 as a parameter.

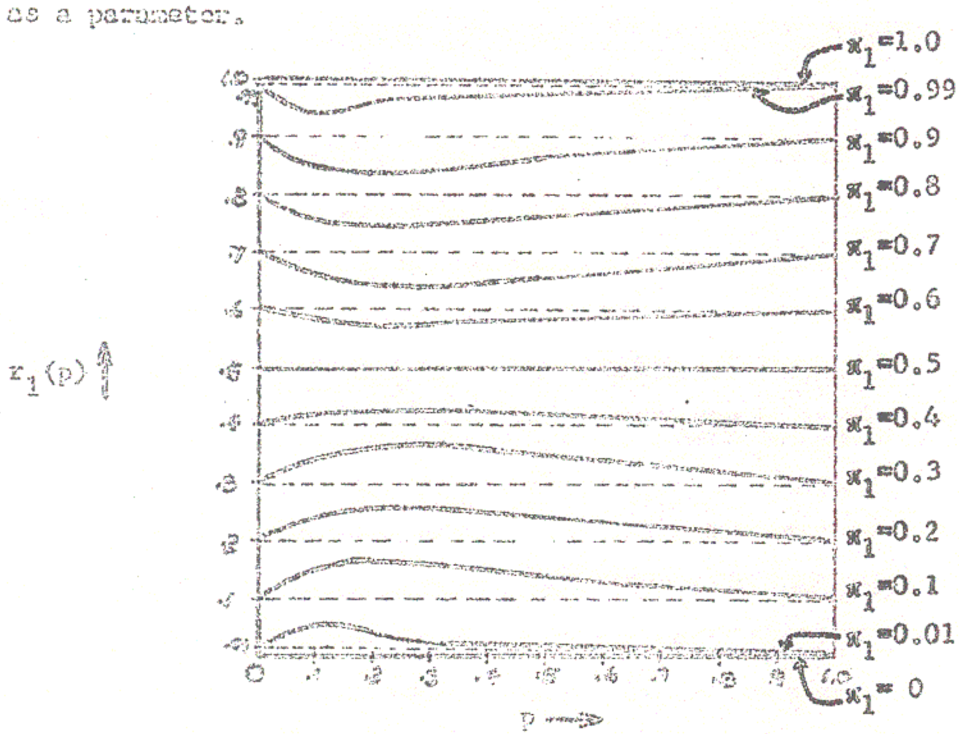

Figure A-1: Variation of r_1 with p.

Note that the variation of r_1 is not too great. This illustrates that although Theorem 2 is written as a negative result, its proof demonstrates a positive result, namely, that the variation of r_1 is not excessive. The arrangement which gives this behavior is one in which the channel capacity is divided between the two channels in proportion to the desired probability of using each channel, and for the discipline followed when a message finds both channels empty, one merely chooses channel i with probability $\varkappa_i$.

Corollary to Theorem 2 - proof:

In proving Theorem 2, it was shown that for $\varkappa_1 = 1/2$, a suitable system could be found to realize this $\varkappa_1$. This result also follows directly from the complete symmetry of the two channels. Similarly, the proof of this corollary follows trivially from recognizing, once again, the complete symmetry of each of the N channels. ▨

Theorem 3 - proof:

The system considered in this theorem satisfies the conditions of the Birth-Death Process examined earlier, with

$$b_n = \lambda$$
$$d_n = \mu(C-\overline{C_n})$$

Thus, by Eqn. (A3), we find

$$P_n = P_0(\lambda/\mu)^n \Big/ \Big[\prod_{i=1}^{n} (C-\overline{C_i}) \Big] \qquad n \geq 1$$

or

$$P_n = P_0 p^n \Big/ \Big[\prod_{i=1}^{n} (1-r_i) \Big] \qquad n \geq 1 \qquad \text{(A11)}$$

where $p = \lambda/\mu C$

$$r_i = \overline{C_i}/C \qquad \text{(A12)}$$

and $P_n = P_0$ for n=0, by definition.

Let us now solve for P_0:

$$\sum_{n=0}^{\infty} P_n = 1 = P_0 \Big[1 + \sum_{n=1}^{\infty} R_n p^n \Big]$$

where

$$R_n = 1 \Big/ \prod_{i=1}^{n} (1-r_i)$$

Thus $$P_0 = 1 / [1 + \sum_{n=1}^{\infty} R_n p^n] \qquad (A13)$$

Now, according to the statement of the theorem, let us form and solve for

$$x = 1 - \sum_{n=0}^{\infty} (\overline{C_n}/C) P_n$$

Noting that $\overline{C_0} = C$ by construction, and using Eqns. (A11) - (A13),

$$x = 1 - P_0 - P_0 \sum_{n=1}^{\infty} r_n R_n p^n$$

$$x = 1 - \frac{1 + \sum_{n=1}^{\infty} r_n R_n p^n}{1 + \sum_{n=1}^{\infty} R_n p^n}$$

$$x = \frac{1 + \sum_{n=1}^{\infty} R_n p^n - 1 - \sum_{n=1}^{\infty} r_n R_n p^n}{1 + \sum_{n=1}^{\infty} R_n p^n}$$

$$x = \frac{\sum_{n=1}^{\infty} R_{n-1} p^n}{1 + \sum_{n=1}^{\infty} R_n p^n}$$

$$x = \frac{p \sum_{n=0}^{\infty} R_n p^n}{1 + \sum_{n=1}^{\infty} R_n p^n}$$

It is important to recognise here that R_0 must be defined as

$$R_0 = (1-r_1)R_1$$

Thus

$R_0 = 1$ (taken now as a definition as well) and so

$$x = p \frac{1 + \sum_{n=1}^{\infty} R_n p^n}{1 + \sum_{n=1}^{\infty} R_n p^n}$$

or $\quad x = p$

which proves the theorem. ∎

Theorem 4 - proof:

The system considered in this theorem satisfies the conditions of the Birth-Death Process examined earlier. However, we have N+1 nodes, and so we must investigate N+1 probability distributions, $P_n^{(m)}$, where $m = 1,2,\ldots,N+1$ and $n = 0,1,2,\ldots$ Let the birth and death rates for node m be $b_n^{(m)}$ and $d_n^{(m)}$ respectively. With this notation, we see that

$$b_n^{(m)} = \lambda_m + \sum_{\substack{j=1 \\ j\neq m}}^{N+1} (\mu C_j/N)\alpha_{jm}[\,1 - \sum_{i=0}^{N-1} P_i^{(j)}(N-i)/N] \qquad n \geq 0$$

$$d_n^{(m)} = \begin{cases} n\mu C_m/N & n = 0,1,\ldots,N \\ \mu C_m & n \geq N \end{cases}$$

An explanation of the $b_n^{(m)}$ is required at this point. λ_m is the input (birth) rate of messages to node m from its external source (by definition). In addition, each of the other N nodes sends messages to node m. Let us consider the j^{th} node's contribution (x_j say) to the input rate of node m ($j \neq m$):

Clearly,

$$x_j dt = Pr[Q_1, Q_2, Q_3] = Pr[Q_1 \mid Q_2, Q_3]\, Pr[Q_2, Q_3]$$

where Q_1 = event that a message on the channel connecting node j to node m completes its transmission to node m in an arbitrary time interval (t, t+dt)

Q_2 = event that an arbitrary message in node j does not have node m for its final address

Q_3 = event that the channel connecting node j to node m is being used

Since Q_2 and Q_3 are independent events, we get

$$x_j dt = Pr[Q_1 \mid Q_2, Q_3] \, Pr[Q_2] \, Pr[Q_3]$$

and for node j,

$$Pr[Q_1 \mid Q_2, Q_3] = Pr[Q_1 \mid Q_3] = (\mu C_j/N)dt$$

$$Pr[Q_2] = \alpha_{jm}$$

$$Pr[Q_3] = 1 - \sum_{i=0}^{N-1} p_i^{(j)}(N-i)/N$$

The summation on j appearing in the expression for $b_n^{(m)}$ merely adds up the contributions to the input (birth) rate of internally routed messages.

Now, according to the definition of $\overline{C_n}$ in Theorem 3, we can apply the same definition to each of the N+1 nodes in the present theorem. Thus, in this case, we see that

$$\overline{C_n}^{(m)} = \begin{cases} [(N-n)/N]C & n \leq N \\ 0 & n \geq N \end{cases}$$

and so we recognize, by application of Theorem 3, that

$$1 - \sum_{i=0}^{N-1} p_i^{(j)}(N-i)/N = p_j$$

where, by definition,

$$\rho_j = \frac{\Gamma_j/\mu}{C_j} = \frac{\text{average input rate to node } j}{\text{maximum output rate from node } j}.$$

$$\text{thus } b_n^{(m)} = \lambda_m + \sum_{\substack{j=1 \\ j\neq m}}^{N+1} \Gamma_j \alpha_{jm}/N \qquad \text{all } n \geq 0$$

But, since $b_n^{(m)}$ is independent of n, it obviously satisfies the definition of Γ_m (= average number of messages per second entering node m). Thus

$$b_n^{(m)} = \Gamma_m = \lambda_m + \sum_{\substack{j=1 \\ j\neq m}}^{N+1} \Gamma_j \alpha_{jm}/N$$

Now, using these birth and death coefficients, we apply eqn. (A3) to get

$$p_n^{(m)} = \begin{cases} P_0^{(m)} (\Gamma_m/\mu C_m)^n N^n/n! & (n = 0, 1, \ldots, N) \\ P_0^{(m)} (\Gamma_m/\mu C_m)^n N^N/N! & n \geq N \end{cases}$$

In this case, it is clear that the steady state is defined only when

$$\rho_m = \Gamma_m/\mu C_m < 1 \qquad \text{for all } m = 1, 2, \ldots, M$$

This completes the proof of the theorem. ■

<u>Theorem 5 - proof:</u>

In order to show that all traffic flowing within the network considered in Theorem 4, is Poisson in nature, it is sufficient to show that

$q(C,t) = Pr$ {a message transmission, in any channel C of the network, is completed in a time interval $(t, t + dt)$, where t is arbitrary}

$= k dt$

where k is a constant.

Let us show this for an arbitrary channel connecting node j (say) to any other node:

$$q(C_j/N,t) = Pr[Q_1 \mid Q_3] \, P[Q_3]$$

where Q_1 and Q_3 are as defined in the proof of Theorem 4. As shown in the proof of Theorem 4,

$$Pr[Q_1 \mid Q_3] \, Pr[Q_3] = (\mu C_j/N) \, [1 - \sum_{i=0}^{N-1} p_i^{(j)} (N-i)/N] \, dt$$

$$= (\mu C_j/N) p_j \, dt$$

and so $q(C_j/N,t) = (\Gamma_j/N) \, dt$

which proves the theorem, and also shows the value of the mean rate for the Poisson traffic to be Γ_j/N. ∎

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